Fokker-Planck-type equations for a simple gas and for a semirelativistic Brownian motion from a relativistic kinetic theory

Guillermo Chacón-Acosta*
Departamento de Física, Universidad Autónoma Metropolitana–Iztapalapa, México D. F. 09340, Mexico

Gilberto M. Kremer†
Departamento de Física, Universidade Federal do Paraná, 81531-990 Curitiba, Brazil
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A covariant Fokker-Planck-type equation for a simple gas and an equation for the Brownian motion are derived from a relativistic kinetic theory based on the Boltzmann equation. For the simple gas the dynamic friction four-vector and the diffusion tensor are identified and written in terms of integrals which take into account the collision processes. In the case of Brownian motion, the Brownian particles are considered as nonrelativistic, whereas the background gas behaves as a relativistic gas. A general expression for the semirelativistic viscous friction coefficient is obtained and the particular case of constant differential cross section is analyzed for which the nonrelativistic and ultrarelativistic limiting cases are calculated.

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I. INTRODUCTION

The problem of Brownian motion played a fundamental role in the early verifications of kinetic theory after the estimation of the Avogadro number by Perrin in 1908 on the basis of Einstein's theory. Brownian motion and diffusion processes have a wide range of applications that goes from chemistry, solid state, quantum physics, plasmas, astronomy and astrophysics to social sciences, life sciences, and biology [1].

Other kinds of applications can be found if we extend the Brownian motion to the regime of the theory of relativity. The relativistic Brownian motion could have important applications in plasma physics, high energy physics [2], astrophysics—for example, in the analysis of the γ ray burst jets where a Brownian motion of the electrons in the electrostatic wave field could be present [3]—in relativistic corrections to the Sunyaev-Zeldovich effect [4], and also in the evolution of dark matter [5].

Recently, several different approaches of a relativistic theory of stochastic processes have been used for the implementation of the Brownian motion concept into the theory of special relativity [6–8], some of them generalized their results to a curved space-time [9], and studied the problem from the point of view of reaching quantum theory with Nelson's method [10], and also from quantum gravity [11,12]. A comparison among different relativistic stochastic processes existing in the literature can be found in [13].

In the works [14,15], Dunkel and Hänggi constructed a relativistic theory for the Brownian motion starting with a relativistic version of the Langevin equation and obtained that the related Fokker-Planck equation can be written as a continuity equation, but with different expressions for the flux density by using different interpretations of the stochastic processes. Moreover, they found that only one of the approaches leads to the equilibrium relativistic Maxwell-Jüttner distribution function, while the others differ through factors that depend on the energy of the particles. In a recent paper by the same authors [16] they have started from a microscopic collision model and constructed a criteria to identify the equilibrium distribution of the particles. First they have used this criteria in the nonrelativistic case and recognized the Maxwellian distribution function, then they applied the same criteria in the relativistic case and, following their previous results, found that the relativistic distribution function that satisfies their criteria, differs from the relativistic Maxwell-Jüttner distribution function by a factor proportional to the inverse of the relativistic kinetic energy. This result does not agree with the one which comes out from the relativistic kinetic theory based on the Boltzmann equation where the only distribution function which implies a vanishing collision term in equilibrium is the Maxwell-Jüttner distribution function (see, e.g., Refs. [17–19]).

It is important to have a theory of Brownian motion founded on kinetic theory, since that theory could give the connection between the fundamental microscopic dynamics and macroscopic schemes which have measurable properties. With such a theory one can obtain the equations that govern the motion, and also calculate the transport coefficients as a function of the properties of the specific system. It is the aim of the present work to investigate a relativistic generalization of the Fokker-Planck equation and of the equation for the Brownian motion within the framework of the relativistic kinetic theory based on the Boltzmann equation.

This work is structured as follows. In Sec. II a covariant version of the Fokker-Planck equation is derived from a relativistic kinetic theory based on the Boltzmann equation and applied to a system with only one constituent. The same assumptions of grazing collisions valid for a nonrelativistic gas (see, e.g., Refs. [20,21]) are also considered here, which consist in small deflections on the scattering angle and small changes in the momentum of the particles at collision. The dynamic friction four-vector and the diffusion tensor are identified and written in terms of integrals which take into

*gca@xanum.uam.mx
†kremer@fisica.ufpr.br

account the collision processes. In Sec. III a mixture of two species is considered where one of the components has a small particle number density and whose particles have a large mass when compared with the other component. Those conditions allow us to call this case a Brownian motion by analogy with the fluctuation process (see, e.g., Ref. [22]). As in the nonrelativistic case (see, e.g., [23]) the former is identified with the Brownian particles and the latter with the background gas. The Brownian particles are considered as nonrelativistic whereas the background gas behaves as a relativistic gas which may alternate from the nonrelativistic limit to the ultrarelativistic limit depending on the ratio between the rest energy of the particles of the gas and of the thermal energy of the mixture. A general expression for the semirelativistic viscous friction coefficient is obtained and the particular case of constant differential cross section is analyzed for which the nonrelativistic and ultrarelativistic limiting cases are calculated. Concluding remarks are given in Sec. IV.

II. RELATIVISTIC FOKKER-PLANCK EQUATION FOR A SIMPLE GAS

Let a particle of a simple relativistic gas be characterized by its rest mass m and by the space-time coordinates $(x^\alpha)=(ct,\mathbf{x})$ and momentum four-vector $(p^\alpha)=(p^0,\mathbf{p})$, where the component of the momentum four-vector p^0 is constrained by $p^0=\sqrt{|\mathbf{p}|+m^2c^2}$. The state of the relativistic gas in the phase space is characterized by the one-particle distribution function $f(x^\alpha,p^\alpha)=f(\mathbf{x},\mathbf{p},t)$, such that $f(\mathbf{x},\mathbf{p},t)d^3x\, d^3p$ gives at time t the number of particles in the volume element d^3x about $\mathbf{x}$ and with momenta in a range d^3p about $\mathbf{p}$.

For an elastic collision of two particles with momentum four-vectors denoted by p^α and p_*^α, the energy-momentum conservation law reads $p^\alpha+p_*^\alpha=p'^\alpha+p_*'^\alpha$, where the quantities p'^α, $p_*'^\alpha$ denote the values of the momentum four-vectors after collision.

The one-particle distribution function satisfies the relativistic Boltzmann equation (see e.g., [18,19])

$$p^\alpha\frac{\partial f}{\partial x^\alpha}+m\frac{\partial fK^\alpha}{\partial p^\alpha}=\int (f_*'f'-f_*f)F\sigma d\Omega\frac{d^3p_*}{p_{*0}}, \tag{1}$$

where the usual abbreviations $f_*'\equiv f(\mathbf{x},\mathbf{p}_*',t)$, $f'\equiv f(\mathbf{x},\mathbf{p}',t)$, $f_*\equiv f(\mathbf{x},\mathbf{p}_*,t)$, and $f\equiv f(\mathbf{x},\mathbf{p},t)$ were introduced. In the above equation, σ denotes a differential cross section, $d\Omega$ an element of solid angle which characterizes the scattering process, K^α the Minkowski external force, and $F=\sqrt{(p_*^\alpha p_\alpha)^2-m^4c^4}$ the so-called invariant flux.

Under the assumption of grazing collisions that could take place in long-range interactions, only small changes in the momentum of the particles occur due to small deflections in scattering angle. Then the collision term on the right-hand side of the Boltzmann equation (1), denoted by $\mathcal{Q}(f,f_*)$, can be approximated as follows.

First new variables are introduced, namely, the total P^α and the relative Q^α momentum four-vectors, defined by

$$P^\alpha=p^\alpha+p_*^\alpha=P'^\alpha,\quad Q^\alpha=p^\alpha-p_*^\alpha,\quad Q'^\alpha=p'^\alpha-p_*'^\alpha. \tag{2}$$

For these quantities the following relationships hold: $P^\alpha Q_\alpha=0$ and $P^2=Q^2+4m^2c^2$, where $P^\alpha P_\alpha=P^2$ and $Q^\alpha Q_\alpha=-Q^2$.

Further, from the energy-momentum conservation law and Eq. (2) one can write the differences between the post- and precollision momentum four-vectors as

$$\Delta p^\alpha\equiv p'^\alpha-p^\alpha=\frac{1}{2}(Q'^\alpha-Q^\alpha)\equiv\frac{1}{2}\Delta Q^\alpha,$$

$$\Delta p_*^\alpha\equiv p_*'^\alpha-p_*^\alpha=-\frac{1}{2}(Q'^\alpha-Q^\alpha)\equiv-\frac{1}{2}\Delta Q^\alpha. \tag{3}$$

Hence for small changes of the momentum of the particles at collision one can expand the one-particle distribution function in Taylor series, which up to the second-order terms reads

$$f(p'^i)\approx f(p^i)+\frac{1}{2}\Delta Q^i\frac{\partial f}{\partial p^i}+\frac{1}{8}\Delta Q^i\Delta Q^j\frac{\partial^2 f}{\partial p^i\,\partial p^j}, \tag{4}$$

with a similar expression for $f(p_*'^i)$, by making the changes $p^i\to p_*^i$ and $\Delta Q^i\to-\Delta Q^i$. In these expressions we take only the first two derivatives because the other contributions are of smaller order thanks to the hypothesis of grazing collisions.

Now it is possible to approximate the collision term of the Boltzmann equation (1) as

$$\mathcal{Q}(f,f_*)=\int\Bigg[\Delta Q^i\frac{\partial}{\partial Q^i}(ff_*)+\frac{1}{2}\Delta Q^i\Delta Q^j\frac{\partial}{\partial Q^i}\frac{\partial}{\partial Q^j}(ff_*)\Bigg]F\sigma d\Omega\frac{d^3p_*}{p_{*0}}, \tag{5}$$

with the help of the relationship

$$\frac{\partial}{\partial Q^i}=\frac{1}{2}\left(\frac{\partial}{\partial p_*^i}-\frac{\partial}{\partial p^i}\right). \tag{6}$$

In order to transform the integral (5), the center-of-mass system is chosen where the spatial components of the total momentum four-vector vanish, i.e.,

$$(P^\alpha)=(P^0,\mathbf{0}),\quad (Q^\alpha)=(0,\mathbf{Q}). \tag{7}$$

Now the element of solid angle in Eq. (5) can be written as $d\Omega=\sin\Theta d\Theta d\Phi$, where Θ and Φ are polar angles of Q'^α with respect to Q^α and such that Θ represents the scattering angle. Further, without loss of generality, Q^α is chosen in the direction of the three axis, so that one can write Q^α and Q'^α as

$$(Q^\alpha)=Q\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\quad (Q'^\alpha)=Q\begin{pmatrix}0\\ \sin\Theta\cos\Phi\\ \sin\Theta\sin\Phi\\ \cos\Theta\end{pmatrix}. \tag{8}$$

By using the above representations it is easy to calculate the following integrals in the variable $0\le\Phi\le2\pi$, yielding

$$\int \Delta Q^i \frac{F}{p_{*0}}\sigma \sin\Theta d\Theta d\Phi = -Q^i Q\Sigma, \tag{9}$$

$$\int \Delta Q^i \Delta Q^j \frac{F}{p_{*0}}\sigma \sin\Theta d\Theta d\Phi = -(Q^2\eta^{ij} + Q^iQ^j)Q\Sigma. \tag{10}$$

Note that that the differential cross section is a function of $\sigma = \sigma(Q,\Theta)$ whereas the invariant flux is given by $F/p_{*0} = Q$. Above Σ denotes the following integral:

$$\Sigma = 2\pi \int (1-\cos\Theta)\sigma \sin\Theta d\Theta, \tag{11}$$

and η^{ij} are the spatial components of the metric tensor $(\eta_{\alpha\beta}) = \mathrm{diag}(1,-1,-1,-1)$. In order to obtain the integral (10) the following approximation was taken into account $\sin^2\Theta \approx 2(1-\cos\Theta)$, since only grazing collisions between the particles with small scattering angles are considered.

By differentiating Eq. (10) with respect to Q^j and considering the relationship $\partial Q/\partial Q^j = -Q_j/Q$, one can obtain the following connection between the integrals (9) and (10):

$$\frac{\partial}{\partial Q^j}\int \Delta Q^i \Delta Q^j Q\sigma d\Omega = 2\int \Delta Q^i Q\sigma d\Omega. \tag{12}$$

Now by making use of Eqs. (6) and (12), the collision integral (5) becomes

$$\mathcal{Q}(f,f_*) = \frac{1}{4}\int \frac{\partial}{\partial p_*^i}\left(\int \frac{\partial (ff_*)}{\partial Q^j}\Delta Q^i \Delta Q^j Q\sigma d\Omega\right)d^3p_* - \frac{1}{4}\frac{\partial}{\partial p^i}\left(\int \frac{\partial (ff_*)}{\partial Q^j}\Delta Q^i \Delta Q^j Q\sigma d\Omega d^3p_*\right). \tag{13}$$

The first term on the right-hand side of the above equation vanishes, since the hypothesis of grazing collisions is used and it is possible to convert—thanks to the divergence theorem—the volume integral in the momentum space into an integral at an infinitely far surface where the distribution functions tend to zero. The second term on the right-hand side can be manipulated by using Eqs. (6) and (12), yielding

$$\mathcal{Q}(f,f_*) = \frac{1}{4}\frac{\partial}{\partial p^i}\left[2\int ff_*\Delta Q^i Q\sigma d\Omega d^3p_* - \frac{1}{2}\int\left(\frac{\partial}{\partial p_*^j} - \frac{\partial}{\partial p^j}\right)\times(ff_*\Delta Q^i\Delta Q^j Q\sigma d\Omega)d^3p_*\right]. \tag{14}$$

By invoking the divergence theorem again and using Eq. (6), it follows that

$$\mathcal{Q}(f,f_*) = -\frac{\partial}{\partial p^i}\left(fA^i - \frac{\partial f D^{ij}}{\partial p^j}\right), \tag{15}$$

where the spatial components of the coefficient of dynamic friction A^i and the diffusion coefficient D^{ij} are given by

$$A^i = \int f_*\Delta p_*^i F\sigma d\Omega \frac{d^3p_*}{p_{*0}}, \tag{16}$$

$$D^{ij} = \frac{1}{2}\int f_*\Delta p_*^i \Delta p_*^j F\sigma d\Omega \frac{d^3p_*}{p_{*0}}. \tag{17}$$

It is important to call attention to the fact that the left-hand side of the Boltzmann equation is a scalar invariant, but the collision term written in the expression (15) is not an invariant. In order to write a covariant form of the Boltzmann equation one has to recall that the above results were obtained by considering the center-of-mass system. In this system $\Delta p_*^0 = 0$, and one can include the zero components

$$A^0 = \int f_*\Delta p_*^0 F\sigma d\Omega \frac{d^3p_*}{p_{*0}}, \tag{18}$$

$$D^{i0} = D^{0i} = \frac{1}{2}\int f_*\Delta p_*^i \Delta p_*^0 F\sigma d\Omega \frac{d^3p_*}{p_{*0}}, \tag{19}$$

$$D^{00} = \frac{1}{2}\int f_*\Delta p_*^0 \Delta p_*^0 F\sigma d\Omega \frac{d^3p_*}{p_{*0}}, \tag{20}$$

into the collision term (15) so that it becomes a scalar invariant. While the components of the coefficient of dynamic friction A^i and of the diffusion coefficient D^{ij} are related with the changes of the momenta Δp_*^i at collision, the components A^0 and D^{00} take into account the energy changes Δp_*^0 and D^{0i} the changes of both Δp_*^0 and Δp_*^i. They are the temporal components of the four-vector A^α and of the four-tensor $D^{\alpha\beta}$.

Hence the relativistic Boltzmann equation (1) reduces to the relativistic Fokker-Planck equation, namely,

$$p^\alpha \frac{\partial f}{\partial x^\alpha} + m\frac{\partial f K^\alpha}{\partial p^\alpha} = -\frac{\partial}{\partial p^\alpha}\left(fA^\alpha - \frac{\partial f D^{\alpha\beta}}{\partial p^\beta}\right). \tag{21}$$

For a spatially homogeneous case without external forces, the Fokker-Planck equation (21) reduces to

$$\frac{p^0}{c}\frac{\partial f}{\partial t} + \frac{\partial \mathcal{F}^\alpha}{\partial p^\alpha} = 0, \tag{22}$$

which represents a continuity equation in momentum space with the particle flux density $\mathcal{F}^\alpha$ given by

$$\mathcal{F}^\alpha = f\left(A^\alpha - \frac{1}{f}\frac{\partial f D^{\alpha\beta}}{\partial p^\beta}\right). \tag{23}$$

The stationary solution of Eq. (22) for the distribution function f is obtained by assuming that the particle flux density vanishes, i.e., $\mathcal{F}^\alpha = 0$. This condition is equivalent to the one that in equilibrium the collision term of the Boltzmann equation vanishes and the distribution function is characterized by the Maxwell-Jüttner distribution [18,19], namely,

$$f \propto \exp\left(-\frac{U_\alpha p^\alpha}{kT}\right), \tag{24}$$

where U_α—such that $U^\alpha U_\alpha = c^2$—is the hydrodynamical four-velocity of the gas.

If we insert Eq. (24) into (23), by considering $\mathcal{F}^\alpha = 0$, it follows that

$$kT\tilde{A}^{\alpha} = -U_{\beta}D^{\alpha\beta}, \quad \text{where } \tilde{A}^{\alpha} = A^{\alpha} - \frac{\partial D^{\alpha\beta}}{\partial p^{\beta}}. \tag{25}$$

The representations of the four-vector $\tilde{A}^{\alpha}$ and of the four-tensor $D^{\alpha\beta}$ in terms of the four-momentum p^{α} read

$$\tilde{A}^{\alpha} = Ap^{\alpha}, \quad D^{\alpha\beta} = -\tilde{D}\eta^{\alpha\beta} - D\frac{p^{\alpha}p^{\beta}}{m^2c^2}, \tag{26}$$

where A, $\tilde{D}$, and D are scalar coefficients and $D^{\alpha}_{\alpha} = -4\tilde{D} - D$.

The insertion of the representations (26) into Eq. (25) leads to

$$\tilde{D} = 0, \quad kT\,A = \frac{p^{\alpha}U_{\alpha}}{m^2c^2}D, \tag{27}$$

i.e., only one among the three coefficients is linearly independent.

Hence, thanks to Eqs. (26) and (27), the spatially homogeneous Fokker-Planck equation (22) reduces to

$$\frac{p^0}{c}\frac{\partial f}{\partial t} + \frac{\partial}{\partial p^{\alpha}}\left[\frac{Dp^{\alpha}}{m^2c^2}\left(f\frac{p^{\alpha}U_{\alpha}}{kT} + p^{\beta}\frac{\partial f}{\partial p^{\beta}}\right)\right] = 0. \tag{28}$$

In a Lorentz rest frame—where $U^{\mu} = (c, \mathbf{0})$—one can obtain from the second Eq. (27)

$$\frac{mkTA}{D} = \sqrt{1 + \frac{|\mathbf{p}|^2}{m^2c^2}} \approx 1 + \frac{v^2}{2c^2} + \cdots, \tag{29}$$

where the leading term, for $v \ll c$, corresponds to the usual relation between the diffusion coefficient and the friction coefficient.

III. RELATIVISTIC EQUATION FOR BROWNIAN MOTION

In this section a mixture of two constituents is considered, where one of the components consist of heavy particles of rest mass $m_{\rm b}$ while the other by light particles of rest mass $m_{\rm g}$, so that $m_{\rm b} \gg m_{\rm g}$. The component with light particles describes a rarefied gas with particle number density much larger than that of the constituent with heavy particles and which characterizes the Brownian particles $n_b \ll n_g$. The gas is supposed to be at equilibrium with a Maxwell-Jüttner distribution function

$$f_{\rm g} = f_{\rm g}^{(0)} = \frac{n_{\rm g}}{4\pi m_{\rm g}^2 ckTK_2(\zeta_{\rm g})}e^{-U_{\alpha}p_{\rm g}^{\alpha}/kT}. \tag{30}$$

Above, $n_{\rm g}$ denotes the particle number density of the gas, c the speed of light, k the Boltzmann constant, whereas T and U^{α} are the temperature and the four-velocity of the mixture, respectively. The symbol $K_2(\zeta_{\rm g})$ refers to a modified Bessel function of second kind and $\zeta_{\rm g} = m_{\rm g}c^2/kT$ gives the ratio between the rest and thermal energies of the gas particles.

The two assumptions above for the Brownian constituent—that it has a small particle number density and particles with large mass with respect to the gas constituent—imply that it may be considered as a nonrelativistic gas with negligible collision term with respect to its particles. Hence the Boltzmann equation for the one-particle distribution function of the Brownian particles can be written as

$$\frac{\partial f_{\rm b}}{\partial t} + v_{\rm b}^i\frac{\partial f_{\rm b}}{\partial x^i} + \frac{\partial(f_{\rm b}F^i)}{\partial p_{\rm b}^i} = \int (f_{\rm g}^{(0)\prime}f_{\rm b}' - f_{\rm g}^{(0)}f_{\rm b})g_{\o}\sigma d\Omega d^3p_{\rm g}. \tag{31}$$

The above equation follows easily from the Boltzmann equation (1) written for a Brownian particle through its multiplication by $c/p_{\rm b}^0$, the introduction of the velocity $v_{\rm b}^i = cp_{\rm b}^i/p_{\rm b}^0$, and of the Møller velocity defined by

$$g_{\o} = \frac{cF}{p_{\rm b}^0p_{\rm g}^0} = \sqrt{(\mathbf{v}_{\rm g} - \mathbf{v}_{\rm b})^2 - \frac{1}{c^2}(\mathbf{v}_{\rm g}\times\mathbf{v}_{\rm b})^2}. \tag{32}$$

Since the collisions of the gas particles with the Brownian particles affect very little the depart from equilibrium of the latter, one can suppose that the distribution function of the Brownian particles can be written as

$$f_{\rm b} = f_{\rm b}^{(0)}h(p_{\rm b}^i), \quad f_{\rm b}^{(0)} = e^{-U_{\alpha}p_{\rm b}^{\alpha}/kT} \tag{33}$$

where $f_{\rm b}^{(0)}$ refers to the exponential factor of a Maxwell-Jüttner distribution function for the Brownian particles and $h(p_{\rm b}^i)$ represents a deviation from this distribution when we assume space homogeneity. Moreover, based on this assumption one can expand the deviation for the post-collision momentum $h(p_{\rm b}'^i)$ in Taylor series and by keeping up to second order terms in the difference $\Delta p_{\rm b}^i = p_{\rm b}'^i - p_{\rm b}^i$, yields

$$h(p_{\rm b}'^i) = h(p_{\rm b}^i) + \Delta p_{\rm b}^i\frac{\partial h}{\partial p_{\rm b}^i} + \frac{1}{2}\Delta p_{\rm b}^i\Delta p_{\rm b}^j\frac{\partial^2 h}{\partial p_{\rm b}^i\,\partial p_{\rm b}^j}. \tag{34}$$

Equations (33) and (34) are introduced into the collision term of the Boltzmann equation (31) so that it can be written as

$$\mathcal{Q}(f_{\rm b}, f_{\rm g}) = f_{\rm b}^{(0)}\mathcal{A}\mathcal{I}, \tag{35}$$

where the relationship $f_{\rm b}'^{(0)}f_{\rm g}'^{(0)} = f_{\rm b}^{(0)}f_{\rm g}^{(0)}$ was used. Above $\mathcal{A} = n_{\rm g}/4\pi m_{\rm g}^2 ckTK_2(\zeta_{\rm g})$ and $\mathcal{I}$ is the following integral

$$\mathcal{I} = \int\left[\Delta p_{\rm b}^i\frac{\partial h}{\partial p_{\rm b}^i} + \frac{1}{2}\Delta p_{\rm b}^i\Delta p_{\rm b}^j\frac{\partial^2 h}{\partial p_{\rm b}^i\,\partial p_{\rm b}^j}\right]e^{-U_{\alpha}p_{\rm g}^{\alpha}/kT}g_{\o}\sigma d\Omega d^3p_{\rm g}. \tag{36}$$

The integral (36) can be transformed as follows. First one can note that for a nonrelativistic Brownian particle $p_{\rm b}^0 = p_{\rm b}'^0$ so that energy conservation law leads to $p_{\rm g}^0 = p_{\rm g}'^0$. Further, one can introduce a relative velocity defined by

$$g^i = \frac{cp_{\rm g}^i}{p_{\rm g}^0} - \frac{cp_{\rm b}^i}{p_{\rm b}^0}, \quad g'^i = \frac{cp_{\rm g}'^i}{p_{\rm g}^0} - \frac{cp_{\rm b}'^i}{p_{\rm b}^0}, \tag{37}$$

such that the difference $\Delta g^i = g'^i - g^i$ can be written, thanks to the momentum conservation law, in terms of the difference Δp^i as follows

$$\Delta g^i = -\frac{c}{p_{\rm g}^0}\left(1 + \frac{p_{\rm g}^0}{p_{\rm b}^0}\right)\Delta p_{\rm b}^i. \tag{38}$$

For a relativistic gas of rest massless particles $|p_{\rm g}^i/p_{\rm g}^0|=1$ whereas $|p_{\rm b}^i/p_{\rm b}^0|=v_b/c$, hence one can approximate the difference given by Eq. (38) as

$$\Delta g^i \approx -\frac{c}{p_{\rm g}^0}\Delta p_{\rm b}^i. \tag{39}$$

The above approximation is also valid for a gas with massive particles, since $m_{\rm b}\gg m_{\rm g}$ and $v_{\rm g}\neq c$ imply that $p_{\rm g}^0/p_{\rm b}^0\ll 1$.

The integral (36) can be written in terms of the difference Δg^i by using Eq. (39). In the resulting equation, two integrations can be performed, the first one can be done by introducing the scattering angle χ and the azimuthal angle ϵ which are the spherical angles of g'^i with respect to g^i. By writing the element of solid angle as $d\Omega=\sin\chi d\chi d\epsilon$ and integrating the differences Δg^i and $\Delta g^i\Delta g^j$ with respect to the azimuthal angle $0\leq\epsilon\leq 2\pi$, yields

$$\int_0^{2\pi}\Delta g^i d\epsilon = -2\pi(1-\cos\chi)g^i, \tag{40}$$

$$\int_0^{2\pi}\Delta g^i\Delta g^j d\epsilon = 2\pi\left(\left[(1-\cos\chi)^2-\tfrac{1}{2}\sin^2\chi\right]g^ig^j - \frac{|\mathbf{g}|^2}{2}\eta^{ij}\sin^2\chi\right), \tag{41}$$

where $|\mathbf{g}|^2=g^kg^k$. For the second integration the element $d^3p_{\rm g}=\sin\phi d\phi d\vartheta|\mathbf{p}_{\rm g}|^2 d|\mathbf{p}_{\rm g}|$ is written in terms of the spherical coordinates $(|\mathbf{p}_{\rm g}|,\vartheta,\phi)$, where ϑ and ϕ are the spherical angles of $p_{\rm g}^i$ with respect to $p_{\rm b}^i$. Hence the integrations of g^i and g^ig^j with respect to ϑ become

$$\int_0^{2\pi} g^i d\vartheta = 2\pi c\left(\frac{|\mathbf{p}_{\rm g}|}{p_{\rm g}^0}\frac{p_{\rm b}^0}{|\mathbf{p}_{\rm b}|}\cos\phi - 1\right)\frac{p_{\rm b}^i}{p_{\rm b}^0}, \tag{42}$$

$$\int_0^{2\pi} g^ig^j d\vartheta = 2\pi c^2\left\{\left[\left(\frac{|\mathbf{p}_{\rm g}|}{p_{\rm g}^0}\frac{p_{\rm b}^0}{|\mathbf{p}_{\rm b}|}\cos\phi - 1\right)^2 - \frac{1}{2}\frac{|\mathbf{p}_{\rm g}|^2}{(p_{\rm g}^0)^2}\frac{(p_{\rm b}^0)^2}{|\mathbf{p}_{\rm b}|^2}\sin^2\phi\right]\frac{p_{\rm b}^ip_{\rm b}^j}{(p_{\rm b}^0)^2} - \frac{1}{2}\frac{|\mathbf{p}_{\rm g}|^2}{(p_{\rm g}^0)^2}\eta^{ij}\sin^2\phi\right\}. \tag{43}$$

The invariant flux $F=\sqrt{(p_{{\rm g}\alpha}p_{\rm b}^\alpha)^2-m_{\rm b}^2m_{\rm g}^2c^4}$ can be approximated for the case of a nonrelativistic Brownian particle by

$$F=\frac{p_{\rm g}^0p_{\rm b}^0g_\varnothing}{c}\approx p_{\rm b}^0|\mathbf{p}_{\rm g}|\left(1-\frac{|\mathbf{p}_{\rm b}|}{p_{\rm b}^0}\frac{p_{\rm g}^0}{|\mathbf{p}_{\rm g}|}\cos\phi\right), \tag{44}$$

by considering terms up to the first order of $|\mathbf{p}_{\rm b}|/p_{\rm b}^0=v_{\rm b}/c$. Note that the above approximation is valid if $p_{\rm g}^0/|\mathbf{p}_{\rm g}|$ does not diverge, but in the following we shall show that after some transformations such a term disappears from the final form of the integral $\mathcal{I}$.

Since the differential cross section is a function of the invariant flux and of the scattering angle it can be approximated by

$$\sigma=\sigma(F,\chi)\approx\sigma(|\mathbf{p}_{\rm g}|,\chi)\left(1-\frac{\partial\ln\sigma}{\partial|\mathbf{p}_{\rm g}|}\frac{|\mathbf{p}_{\rm b}|}{p_{\rm b}^0}p_{\rm g}^0\cos\phi\right). \tag{45}$$

Now by using Eqs. (39) through (45) in Eq. (36) and integrating the resulting equation with respect to the angle $0\leq\phi\leq\pi$ it follows that

$$\mathcal{I}=4\pi^2c\int e^{-U_\alpha p_{\rm g}^\alpha/kT}|\mathbf{p}_{\rm g}|^3\sigma(|\mathbf{p}_{\rm g}|,\chi)(1-\cos\chi)\sin\chi \times\left\{-2p_{\rm g}^0\frac{\partial h}{\partial p_{\rm b}^i}\frac{p_{\rm b}^i}{p_{\rm b}^0}\left[1+\frac{1}{3}\left(1+|\mathbf{p}_{\rm g}|\frac{\partial\ln\sigma}{\partial|\mathbf{p}_{\rm g}|}\right)\right] - \frac{2}{3}(p_{\rm g}^0)^2\frac{\partial^2 h}{\partial p_{\rm b}^i\partial p_{\rm b}^j}\eta^{ij}\frac{|\mathbf{p}_{\rm g}|^2}{(p_{\rm g}^0)^2}\right\}d\chi\frac{d|\mathbf{p}_{\rm g}|}{p_{\rm g}^0}. \tag{46}$$

In order to integrate by parts the derivative of the cross section σ with respect to $|\mathbf{p}_{\rm g}|$ a Lorentz rest frame in which $(U^\alpha)=(c,\mathbf{0})$ is chosen, so that one can obtain

$$\int e^{-cp_{\rm g}^0/kT}|\mathbf{p}_{\rm g}|^4\frac{\partial\sigma}{\partial|\mathbf{p}_{\rm g}|}d|\mathbf{p}_{\rm g}| = -\int e^{-cp_{\rm g}^0/kT}\left(4|\mathbf{p}_{\rm g}|^3 - \frac{c}{kTp_{\rm g}^0}|\mathbf{p}_{\rm g}|^5\right)\sigma(|\mathbf{p}_{\rm g}|,\chi)d|\mathbf{p}_{\rm g}|. \tag{47}$$

Now the substitution of the above result into Eq. (46) leads to

$$\mathcal{I}=-\frac{8}{3}\pi^2c\int e^{-cp_{\rm g}^0/kT}|\mathbf{p}_{\rm g}|^5\sigma(|\mathbf{p}_{\rm g}|,\chi)(1-\cos\chi)\sin\chi \times\left(\frac{cp_{\rm b}^i}{kTp_{\rm b}^0}\frac{\partial h}{\partial p_{\rm b}^i}+\eta^{ij}\frac{\partial^2 h}{\partial p_{\rm b}^i\partial p_{\rm b}^j}\right)d\chi\frac{d|\mathbf{p}_{\rm g}|}{p_{\rm g}^0}. \tag{48}$$

The final form of the collision term (35) in terms of the distribution function of the Brownian particle can be obtained from Eq. (48) by using the representation (33), where the differentiation was also performed in a Lorentz rest reference frame, namely,

$$\mathcal{Q}(f_{\rm b},f_{\rm g})=\eta\left(m_{\rm b}kT\frac{\partial^2 f_{\rm b}}{\partial p_{\rm b}^i\partial p_{\rm b}^j}+\frac{\partial f_{\rm b}p_{\rm b}^i}{\partial p_{\rm b}^i}\right), \tag{49}$$

so that the Boltzmann equation for the Brownian particle can be written as

$$\frac{\partial f_{\rm b}}{\partial t}+\frac{\partial(f_{\rm b}F^i)}{\partial p_{\rm b}^i}=\eta\left(m_{\rm b}kT\frac{\partial^2 f_{\rm b}}{\partial p_{\rm b}^i\partial p_{\rm b}^j}+\frac{\partial f_{\rm b}p_{\rm b}^i}{\partial p_{\rm b}^i}\right). \tag{50}$$

In Eqs. (49) and (50) η is the so-called viscous friction coefficient for a semirelativistic case and it is given by

$$\eta=\frac{2}{3}\frac{n_{\rm g}\pi}{m_{\rm b}(m_{\rm g}kT)^2K_2(\zeta_{\rm g})}\times\int\sigma(|\mathbf{p}_{\rm g}|,\chi)(1-\cos\chi)\sin\chi d\chi e^{-cp_{\rm g}^0/kT}|\mathbf{p}_{\rm g}|^5\frac{d|\mathbf{p}_{\rm g}|}{p_{\rm g}^0}. \tag{51}$$

Equation (50) is the extension to a background gas of relativistic particles of the Fokker-Planck-type equation for Brownian motion which was analyzed by Chandrasekhar

[24], Green [25], and Wang Chang and Uhlenbeck [23] for the case of a nonrelativistic background gas. The relativistic corrections appear here in the viscous friction coefficient.

In order to evaluate the integral for the viscous friction coefficient (51) it is necessary to know the differential cross section σ. For a constant differential cross section the integration in the angle $0 \leq \chi \leq \pi$ is straightforward and by introducing a new variable $y = p_g^0/m_g c$ so that $|\mathbf{p}_g| = m_g c\sqrt{y^2-1}$, the friction viscous coefficient becomes

$$\eta = \frac{4\pi}{3}\frac{n_g \sigma}{m_b(m_g kT)^2 K_2(\zeta_g)}\int_1^\infty e^{-\zeta_g y}(y^2-1)^2 dy, \qquad (52)$$

which leads through integration to the final form of the viscous friction coefficient of the Brownian particles in a background gas of relativistic particles,

$$\eta = \frac{32\pi n_g \sigma kT}{3m_b c K_2(\zeta_g)}\frac{e^{-\zeta_g}}{\zeta_g^2}(3+3\zeta_g+\zeta_g^2). \qquad (53)$$

For low temperatures the rest energy of the background gas $m_g c^2$ is much larger than the thermal energy kT, so that $\zeta_g \gg 1$ and the semirelativistic viscous friction coefficient can be approximated by

$$\eta = \frac{32 n_g \sigma}{3m_b}\sqrt{2m_g \pi kT}\left[1 + \frac{9}{8\zeta_g} + \frac{9}{128\zeta_g^2} + \cdots\right]. \qquad (54)$$

By using the hard-sphere cross section of a nonrelativistic gas $\sigma = d^2/4$—where d denotes the diameter of a particle—the first term in the expression (54) reduces to the result obtained by Wang Chang and Uhlenbeck [23]. The other terms in the series are related with relativistic corrections.

For very high temperatures the parameter $\zeta_g \ll 1$ and this condition characterizes the ultrarelativistic regime. In this case the approximation for the viscous friction coefficient reads

$$\eta = \frac{16\pi n_g \sigma kT}{m_b c}\left\{1 + \frac{\zeta_g^2}{12} + \frac{\zeta_g^4}{64}\left[1 + 4\ln\left(\frac{\zeta_g}{2}\right) + 4\gamma\right] + \cdots\right\}, \qquad (55)$$

where $\gamma = 0.577\,215\,664\ldots$ is the Euler constant. In this case the first term is the leading one and the others are corrections to the former.

IV. CONCLUDING REMARKS

In this work we obtain a manifestly covariant relativistic Fokker-Planck-type equation for the evolution of the distribution function of a simple gas under the assumption of grazing collisions. We also obtain a semirelativistic Fokker-Planck equation for the Brownian motion.

Although the Fokker-Planck equation (21) and the equation of the Brownian motion (50) are alike—at least in the nonrelativistic limit—one has to be very careful with the temptation to obtain the latter from the former. One can write a Fokker-Plack equation such as the one given in Eq. (21) for a mixture of two constituents where one of the constituents has a small particle number density with respect to the other, so that only collisions between dissimilar particles are taken into account. However, in Sec. II only grazing collisions between the particles were taken into account, while in the case analyzed in Sec. III the collisions between the particles of the background gas and the Brownian particles were not restricted to grazing collisions.

The extension of the Fokker-Planck-type equation for a simple gas in the presence of a gravitational field is straightforward, since the Boltzmann equation in gravitational fields is written as (see, e.g., [18])

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu}p^\mu p^\nu \frac{\partial f}{\partial p^\sigma} = \int (f'_* f' - f_* f)F\sigma d\Omega \sqrt{g}\frac{d^3 p_*}{p_{*0}}. \qquad (56)$$

Here $\sqrt{g} = \sqrt{-\det(g_{\mu\nu})}$ where $g_{\mu\nu}$ denotes the metric tensor and $\Gamma^\sigma_{\mu\nu}$ is the Cristoffel symbol. Indeed, the Fokker-Planck equation for this case becomes

$$p^\mu \frac{\partial f}{\partial x^\mu} - \Gamma^\sigma_{\mu\nu}p^\mu p^\nu \frac{\partial f}{\partial p^\sigma} = -\frac{\partial}{\partial p^\mu}\left(fA^\mu - \frac{\partial f D^{\mu\nu}}{\partial p^\nu}\right), \qquad (57)$$

where the dynamic friction four-vector A^μ and the diffusion tensor $D^{\mu\nu}$ are given by

$$A^\mu = \int f_* \Delta p_*^\mu F\sigma d\Omega \sqrt{g}\frac{d^3 p_*}{p_{*0}}, \qquad (58)$$

$$D^{\mu\nu} = \frac{1}{2}\int f_* \Delta p_*^\mu \Delta p_*^\nu F\sigma d\Omega \sqrt{g}\frac{d^3 p_*}{p_{*0}}. \qquad (59)$$

In Sec. III we made an analogy with Brownian motion in the relativistic case, that analogy was made through scattering assumptions based on specific properties such as mass and density ratios of the constituents of the system. These assumptions allowed us to made a physical analogy with Brownian motion and none of the mathematical stochastic properties of Brownian motion were used. In fact the evolution in this version of Brownian motion is provided by the collisions while in other works [6–8,14,15] is given by a stochastic term.

Recently in [26] it is shown that a modified Maxwell-Jüttner distribution could be obtained from a modified principle of maximum entropy that take into account the properties of Lorentz group. This is a hint that the relativistic Boltzmann equation might require a slight modification. Indeed the relativistic kinetic theory considered here is not the only version, there are other proposals such as [27,28] which try to improve some points of the theory.

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